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On the steady flow produced by a rotating disc with either surface suction or injection

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SUMMARY

A series solution is presented for the steady, laminar flow produced by a rotating disc. The series consists solely of exponential terms with negative exponents. It is shown that this approach yields uniformly valid solutions of high accuracy for all cases of suction and for low values of injection at the disc surface. The radius of convergence of the series is determined. For those injection cases for which the direct series method fails, an integral method is presented which is based on the properties of the series. The latter method consists of obtaining differential equations which represent the behaviour of the sums of the series. This method allows the solution of the governing differential equations as an initial value problem.

1. Introduction

The flow induced by the steady rotation of a flat impermeable disc in a fluid which has no rotation far from the disc has received considerable attention over the years. Von Kármán [1] noted first that the governing Navier–Stokes equations reduce to self-similar forms and obtained an approximate solution using his momentum integral method. Cochran [2] obtained a more accurate numerical solution using the approach used by Blasius [3] in his study of the laminar boundary layer on a flat plate in a uniform stream. This approach was to develop two analytical series solutions, one valid far from the disc and the other valid near the disc, the two being matched by numerical calculation at an intermediate position in the flow. The interesting point, so far as the present investigation is concerned, is that the three series used to describe the flow far from the disc have relatively simple forms, consisting solely of exponential terms with negative exponents.

More recently the rotating disc problem has been extended to include uniform suction or injection at the disc surface. Stuart [4] considered the situation in which there is strong suction and obtained an analytical series solution for this case. Kuiken [5] examined the case of strong injection and found that the flow divides into inner and outer layers, the inner layer being shown to be inviscid to first order. Kuiken [5] then obtained higher order (viscous) terms using the method of matched asymptotic expansions. Direct numerical integration of the governing ordinary differential equations, covering cases of both suction and injection, was carried out by Sparrow and Gregg [6].

It is relevant to turn briefly to another but related problem, that of the steady laminar boundary-layer motion on a moving flat conveyor belt. The fluid far from the moving belt is nominally at rest and Ackroyd [7] showed that the solution to the governing self-similar form of the momentum equation can be obtained by the use of a series solution which is uniformly valid throughout the boundary-layer flow. This series solution consists solely of exponential terms with negative exponents. An identical approach was used by Lock [8] to obtain part of the solution to the problem of the laminar merging of two uniform parallel streams. The conveyor belt problem was extended by Samuel and Hall [9] to include both suction and injection effects at the belt surface. Here, a simple modification of the exponential series was found to accommodate all suction cases and a range of injection cases. Samuel and Hall [9] also discussed the question of the convergence of Ackroyd's [7] series and developed a rather ingenious method for locating the position of the singularity of the series and hence its radius of convergence. The method was based on the fact that the sum of the exponential series must satisfy the original ordinary differential equation governing the boundary-layer flow. The method has additional value in that it allows solutions for all cases of suction and injection to be obtained by the single integration of the ordinary differential equation representing the sum of the series. Samuel and Hall [9] showed further that not only was the latter differential equation capable of being posed as an initial value problem but that its integration produced solutions for injection cases beyond the radius of convergence of the series upon which it was based. The method produced a most useful example of analytic continuation.

The above-noted features of the conveyor belt problem led the present author to the belief that many more viscous flows which are induced by the steady motion of a surface in a fluid nominally at rest might be described by such exponential series solutions. The argument in support of this belief can be expressed as follows. Provided the flow exhibits self-similar properties, the governing ordinary differential equations for the viscous flow should always yield a behaviour at the outer region which has an exponential series form. Whether or not such exponential series provide uniformly valid solutions throughout the whole of the viscous flow will depend then on the convergence properties of the particular series. The approach of Samuel and Hall [9] provides a rapid and accurate method for assessing the convergence properties of such series. So far as the rotating disc problem is concerned, the exponential series of Cochran [2] provided encouragement for this belief. Initial calculations for the rotating disc problem, described in Sections 2 and 3 of the present paper, quickly confirmed that such series provide uniformly valid solutions for all suction and some injection cases. It was at this stage that the work of Fettis [10] and Benton [11] came to light. It was found that an identical approach to the present one had been used by Benton [11] for the impermeable disc, Benton [11], in turn, having based his approach on the earlier but slightly different approach of Fettis [10].

Fettis [10] had been concerned largely with the problem of Bödewadt [12] in which solid body rotation occurs in the flow far from the rotating disc. However, Fettis [10] showed that when no solid body rotation occurs, his approach produced a solution consisting solely of exponential terms with negative exponents. He went on to discuss very briefly certain cases in which suction occurs at the disc surface. His method consisted of the well-known procedure for finding the asymptotic behaviour of the governing differential equations as the outer edge of the viscous flow is approached; the equations are expanded in terms of a convenient parameter. In this way, an infinite sequence of ordinary linear differential equations was generated and Fettis [10] solved the first eight of these analytically. Because he believed that this approach provided solutions which are uniformly valid throughout the flow, he evaluated the integration constants introduced at each successive term of his solution by the use of the boundary conditions at the disc surface. Now, successive analytical solutions of the linear ordinary differential equations introduce successively higher powers of exponential terms together with further terms of lower powers. Consequently, it follows that Fettis's [10] method provides only an approximation to the coefficient of any given power of an exponential term. The sheer algebraic labour in obtaining analytical solutions for the successively more lengthy ordinary differential equations militates against the use of the method if high numerical accuracy is required. This difficulty was overcome by Benton [11], using an infinite series of exponential terms. As Benton [11] showed, substitution of this series into the governing differential equations allows recurrence relations to be established between successive coefficients in the series. Thus, in principle, the series can be summed to any desired degree of accuracy. This approach is identical to that adopted by Lock [8] and Ackroyd [7]. Benton [11] had been concerned mainly with the impulsive rotation of a disc from rest and obtained the steady flow solution (i.e. von Kármán's [1] case), using this method, in order to provide an accurate check on the development of his unsteady solution at large time after the initiation of the disc motion. Thus, the present investigation can be seen as an extension of Fettis's [10] and Benton's [11] work to include a more general discussion of suction and injection cases, together with an extended discussion of the problems of convergence.

In Section 3 of the present paper it is shown that the series solution is in complete agreement with Stuart's [4] case of strong suction. In Section 4 the method of Samuel and Hall [9] is adapted to the present problem and the radii of convergence of the two series for various values of the suction or injection parameter are obtained. It is also shown that the method allows solutions to be obtained beyond the radius of convergence of these series. This, in itself, appears to be of no great benefit, since the original ordinary differential equations for the flow can be integrated numerically for those cases in which the original series method falls outside the range in which series convergence is assured. However, the method has the useful feature that the problem of integrating the ordinary differential equations representing the sums of the two series is recast as an initial value problem whereas originally it was posed as a two-point boundary value problem.

2. Similarity equations and series solution

In considering the fluid motion caused by the rotating disc, we use cylindrical polar coordinates r, ϕ, z and denote the corresponding velocity components by u, v, w. The disc surface occupies the plane $z = 0$ and rotates about the z-axis with constant angular velocity ω. Following von Kármán [1], we write

$$u = r\omega F(\zeta),\ v = r\omega G(\zeta),\ w = \sqrt{\nu\omega}H(\zeta),\ \zeta = z\sqrt{\omega/\nu}. \tag{1}$$

We denote the difference between the actual and the hydrostatic pressures by p and write

$$p = \rho\nu\omega P(\zeta). \tag{2}$$

In terms of the non-dimensional variables F, G, H and P defined in equations (1) and (2), the continuity and Navier–Stokes equations in cylindrical polar form become

$$2F + H' = 0,\ F^2 - G^2 + F'H = F'',$$
$$2FG + HG' = G'',\ H'' - HH' = P'. \tag{3}$$

Dashes denote differentiation with respect to the independent variable ζ.

We suppose that the disc is porous and that either uniform suction or injection occurs there such that

$$w(0) = -\sqrt{\nu\omega}A. \tag{4}$$

The parameter A is a constant and $A > 0$ corresponds to the suction case. The boundary conditions attached to equations (3) are, therefore,

$$F(0) = 0,\quad G(0) = 1,\quad H(0) = -A,$$
$$F(\infty) = 0,\quad G(\infty) = 0,\quad P(\infty) = 0. \tag{5}$$

It follows from equations (3) and (5) that as $\zeta \to \infty$, $H \to -c$ $(c > 0)$, so that far from the disc a uniform flow toward the disc is induced by the disc motion. The unknown constant, c, is a function of A only. Use of the first and the last of the equations (3), together with the boundary conditions (5), yields the result for the non-dimensional pressure field

$$P = \frac{c^2}{2} - \left(2F + \frac{H^2}{2}\right),$$

so that (6)

$$P(0) = \tfrac{1}{2}(c^2 - A^2).$$

Cochran's [2] work suggests that, for large values of ζ, the solutions for F, G and H can be written as the following infinite series:

$$F = \sum_1^\infty a_n e^{-nc\zeta},\quad G = \sum_1^\infty b_n e^{-nc\zeta},\quad H = -c + \sum_1^\infty c_n e^{-nc\zeta}. \tag{7}$$

Substitution of the above series into the first three of equations (3) yields

$$c_n = \frac{2a_n}{nc}, \tag{8}$$

together with the recurrence relations for a_n and b_n for $n > 1$,

$$a_n = \frac{1}{c^2 n(n-1)} \sum_{q=1}^{n-1} \left(\frac{3q - 2n}{q} a_q a_{n-q} - b_q b_{n-q}\right),$$
$$b_n = \frac{2}{c^2 n(n-1)} \sum_{q=1}^{n-1} \frac{2q - n}{q} a_q b_{n-q}. \tag{9}$$

For future reference, a_n and b_n for $2 \leqslant n \leqslant 6$ are given in Table 1 in terms of a_1, b_1 and c. The values of a_n and b_n for $2 \leqslant n \leqslant 4$ in this table correspond to those obtained by Cochran [2].

TABLE 1

Coefficients of a_n, b_n in terms of a_1, b_1 and c

n	a_n	b_n
2	$-\frac{1}{2c^2}(a_1^2+b_1^2)$	0
3	$\frac{a_1}{4c^4}(a_1^2+b_1^2)$	$-\frac{b_1}{12c^4}(a_1^2+b_1^2)$
4	$-\frac{(a_1^2+b_1^2)}{144c^6}(17a_1^2+b_1^2)$	$\frac{a_1 b_1}{18c^6}(a_1^2+b_1^2)$
5	$\frac{a_1(a_1^2+b_1^2)}{1152c^8}(61a_1^2+13b_1^2)$	$-\frac{b_1(a_1^2+b_1^2)}{1920c^8}(53a_1^2+5b_1^2)$
6	$-\frac{(a_1^2+b_1^2)}{480c^{10}}\left(\frac{219}{20}a_1^4+\frac{146}{45}a_1^2b_1^2+\frac{7}{36}b_1^4\right)$	$\frac{a_1b_1(a_1^2+b_1^2)}{5400c^{10}}(65a_1^2+17b_1^2)$

It is our intention to demonstrate that the series (7) provide uniformly valid solutions to the equations (3) (with boundary conditions (5)) for a useful range of values of A. As has been remarked earlier, Benton [11], working in terms of b_n/c^2 and c_n/c, has established this point already for the case $A = 0$. Furthermore, Fettis's [10] less detailed calculations indicate that the series (7) might be applicable to suction cases ($A > 0$). Therefore, for the range of A for which the series (7) remain uniformly valid, application of the three boundary conditions at $\zeta = 0$ (equations (5)) allows the simultaneous determination of a_1, b_1 and c. We note that these three parameters are functions of A only. Thus, from equations (5) and (7) we obtain

$$\sum_1^\infty a_n = 0, \quad \left(\sum_1^\infty b_n\right) - 1 = 0, \quad \left(\sum_1^\infty \frac{a_n}{n}\right) - \frac{c(c-A)}{2} = 0. \tag{10}$$

For certain circumstances, rather more useful forms of the series (7) can be developed by writing

$$a_n = c^2\left(\frac{a_1}{c^2}\right)^n A_n, \quad b_n = c^2\left(\frac{b_1}{c^2}\right)^n B_n,$$
$$Z = \frac{a_1}{c^2}\,\mathrm{e}^{-c\zeta}, \quad K = \frac{a_1}{b_1}. \tag{11}$$

Thus

$$F = c^2 f(Z;K), \quad G = c^2 g(Z;K), \quad H = -c(1 - 2h(Z;K)),$$

where (12)

$$f(Z;K) = \sum_1^\infty A_n Z^n, \quad g(Z;K) = \sum_1^\infty B_n\left(\frac{Z}{K}\right)^n, \quad h(Z;K) = \sum_1^\infty \frac{A_n}{n} Z^n.$$

Note that the dependence of f, g and h on K is a parametric dependence only and that K depends solely on A. The recurrence relations for A_n and B_n can be obtained directly from equations (9) and (11) but now

$$A_1 = B_1 = 1,$$

with

$$A_n = \frac{1}{n(n-1)} \sum_{q=1}^{n-1} \left(\frac{3q-2n}{q} A_q A_{n-q} - \frac{B_q B_{n-q}}{K^n} \right), \tag{13}$$

$$B_n = \frac{2}{n(n-1)} \sum_{q=1}^{n-1} \frac{2q-n}{q} K^q A_q B_{n-q}.$$

Equations (10) become

$$\sum_1^\infty A_n Z_0^n = 0, \quad \left(\sum_1^\infty B_n \left(\frac{Z_0}{K} \right)^n \right) - \frac{1}{c^2} = 0, \quad \left(\sum_1^\infty \frac{A_n}{n} Z_0^n \right) - \frac{1-(A/c)}{2} = 0, \tag{14}$$

where $Z_0 = a_1/c^2$ is the value of Z at $\zeta = 0$.

Two alternative methods were employed in the numerical determination of a_1, b_1 and c in which either equations (10) or equations (14) were used. In both methods, the relevant series were terminated at the m-th term, the value of m being subsequently increased until the required accuracy was achieved. In both methods it is necessary to employ an iterative procedure in order to search for the correct values of the relevant parameters. Since most of the terms in the series turn out to alternate in sign, it was found to be useful to use Aitken's δ^2 method with each series in order to accelerate convergence. In all cases, Newton's method was found to be quite adequate for use in the iterations.

In the first method, for a selected value of A, equations (10) were employed in order to search for those values of a_1, b_1 and c which provided values for the left hand sides of equations (10) which were all less than 10^{-10}. The value of m was increased progressively until Aitken's corrections for the three series (10) had decayed to less than 10^{-10}. At this stage, it was seen that the values of a_1, b_1 and c had ceased to vary with m up to and including the ninth decimal place.

In the second method, employing equations (14), selection of a value for K allowed Z_0 to be determined by iteration on the first of equations (14) only. Note that, according to equations (13), A_n depend directly on K. Values of c and A then followed immediately from the second and third of equations (14). Iteration was then performed on the value of K until the desired value of A had been achieved.

Despite the apparent attraction of the second method, in practice there was found to be little to choose between the two methods and they produced results which were virtually indistinguishable. Finally, both methods were used to calculate $F'(0)$ and $G'(0)$ from the relations

$$F'(0) = -c \sum_1^\infty n a_n = -c^3 Z_0 \sum_1^\infty n A_n Z_0^{n-1},$$

$$G'(0) = -c \sum_1^\infty n b_n = -c^3 \frac{Z_0}{K} \sum_1^\infty n B_n \left(\frac{Z_0}{K} \right)^{n-1}. \tag{15}$$

Aitken's δ^2 method was used here once again to accelerate convergence. In most cases, it was found to be necessary to increase m beyond that value required for the determination of a_1, b_1 and c in order to ensure that Aitken's corrections in the series for $F'(0)$ and $G'(0)$ had decayed to less than 10^{-10}.

The values of a_1, b_1, c, $F'(0)$, $G'(0)$ and $P(0)$ (from equations (6)), determined in the above manner for a particular value of A, are reproduced in Table 2. The approximate values of m necessary to satisfy the accuracy requirements are indicated in the table. It will be seen that, from Table 2, whereas suction cases ($A > 0$) require remarkably small values of m, m becomes prohibitively large as the injection rate increases ($-A$ increases). In the latter cases, series convergence became exceedingly slow.

The results given in Table 2 confirm the values of the relevant parameters calculated by Benton [11], who quotes values to six figure accuracy for the case $A = 0$. In the case of the calculations briefly noted by Fettis [10] for suction cases, it was found that the values for c quoted there are in error after the third decimal place. However, it should be noted that Fettis [10] terminated his series at $m = 8$ and that, by the nature of his approach, his determination of a_n and b_n for $n \leqslant 8$ is necessarily incomplete. As for the results provided to four figures by Sparrow and Gregg [6], obtained by the direct numerical integration of equations (3), where comparison can be made with the present calculations (i.e. $A = 3, 2, 0, -0.5$) complete agreement occurs.

3. The case of strong suction

The behaviour of $F(\zeta)$, $G(\zeta)$ and $H(\zeta)$ for the situation in which $A \to \infty$ (the strong suction case) was examined in detail by Stuart [4]. As this limit is approached, Stuart [4] found that the viscous region in contact with the disc becomes progressively thinner and that $H \sim -A$. In the present context, from equations (7), this suggests that $c \sim A$. Furthermore, the results for m in Table 2 suggest that for $A \to \infty$ the series (10) become completely dominated by their leading terms. Thus we might expect that $b_1 \to 1$, $a_1 \to 0$ as this limit is approached. In carrying out a detailed comparison between Stuart's [4] series solution and the present

TABLE 2

Results for a_1, b_1, c, $F'(0)$, $G'(0)$, $P(0)$, m obtained from equations (10) *and* (14)

A	a_1	b_1	c	$F'(0)$	$-G'(0)$	$P(0)$	m (approximate)
3	0.0550069888	1.001006605	3.018208248	0.1655778258	3.0121418847	0.0547905145	8
2	0.1192886125	1.004699961	2.057722613	0.2424161848	2.0385268141	0.1171111765	10
1	0.3391879035	1.035820071	1.260553113	0.3895662270	1.1752208122	0.2944970754	20
0	0.9248635311	1.202211750	0.8844741102	0.5102326189	0.6159220144	0.3911472258	50
−0.2	1.098698978	1.261946133	0.8484350265	0.5162650179	0.5379560054	0.3399209971	90
−0.4	1.296042868	1.331974733	0.8195304869	0.5164467222	0.4683615657	0.2558151096	200
−0.5	1.404491564	1.371093196	0.8072116676	0.5145662968	0.4364319473	0.2007953382	400
−0.55	1.461349736	1.391726933	0.8015071103	0.5131762756	0.4211439279	0.1699568239	1000

approach, it is rather more useful to use the series in the forms of equations (10) rather than equations (14). It can be shown that for $A \to \infty$, $a_1 \sim 1/(2A^2)$ so that we can write

$$a_1 \simeq S_1/(2A^2), \quad b_1 \simeq S_2, \quad c \simeq AS_3. \tag{16}$$

By inspection of the coefficients a_n and b_n (see Table 1) in the series (10), we see that the series S_i ($i = 1, 2, 3$) must take the forms

$$S_i \simeq 1 + \frac{\alpha_i}{A^4} + \frac{\beta_i}{A^8} + \frac{\gamma_i}{A^{12}} + O(A^{-16}). \tag{17}$$

The coefficients α_i, β_i and γ_i can be determined by the substitution of the series (17) into the equations (10), using the values for a_n, b_n given in Table 1. The values for α_i, β_i and γ_i determined in this manner are given in Table 3. Use of these results, in conjunction with the results for a_n, b_n in Table 1 and the series (7), yields expressions for $F(\zeta)$, $G(\zeta)$ and $H(\zeta)$ which are identical to those obtained by Stuart [4] (his equations (3.10), (3.11) and (3.12)). In order to reproduce Stuart's [4] results, it is necessary to know a_n for $n \leqslant 6$ and it is for this reason that the results given in Table 1 have been determined to that level. For the same reason, it is necessary to know γ_3 only so that the determination of γ_1 and γ_2 can be omitted.

4. Application of the method of Samuel and Hall

As noted in Section 2, the series (10) experience convergence difficulties at quite modest negative values of A (the injection case). Similar difficulties were encountered by Samuel and Hall [9] in their related study of the laminar boundary layer flow on a porous conveyor belt. As has been mentioned earlier, the method developed by Samuel and Hall [9] for dealing with this difficulty consists of obtaining differential equations which represent the behaviour of the sums of the series. In this particular application, it is convenient to use the series expressed in the forms given in equations (12).

Substitution of the equations (12) into the differential equations (3) yields the following five simultaneous first order differential equations in terms of the independent variable Z:

TABLE 3

Coefficients α_i, β_i, γ_i

i	α_i	β_i	γ_i
1	$-\frac{59}{72}$	$\frac{28369}{17280}$	–
2	$\frac{1}{12}$	$-\frac{173}{1152}$	–
3	$\frac{1}{2}$	$-\frac{201}{288}$	$\frac{21023}{12960}$

$$\frac{df}{dZ} = U, \qquad f(0;K) = 0,$$

$$\frac{dg}{dZ} = V, \qquad g(0;K) = 0,$$

$$\frac{dh}{dZ} = \frac{f}{Z}, \qquad h(0;K) = 0, \tag{18}$$

$$\frac{dU}{dZ} = \frac{1}{Z^2}(f^2 - g^2 - 2ZhU), \qquad U(0;K) = 1,$$

$$\frac{dV}{dZ} = \frac{2}{Z^2}(fg - ZhV), \qquad V(0;K) = \frac{1}{K}.$$

The last two boundary conditions are obtained from the series (12) and the additional use of the recurrence relations (13) establishes that at

$$Z = 0, \quad \frac{dh}{dZ} = 1, \quad \frac{dU}{dZ} = -\left(1 + \frac{1}{K^2}\right), \quad \frac{dV}{dZ} = 0. \tag{19}$$

Consequently, for a specified value of K, numerical integration of the equations (18) (posed as an initial value problem) can proceed from the point $Z = 0$ in the direction of increasing Z. Since $Z = 0$ corresponds to the outer edge of the viscous layer, integration in this direction corresponds to integration toward the disc surface. A typical result obtained for $f(Z;K)$, for $K = 1.5$, is shown in Fig. 1. The disc surface is located at $Z = Z_0$, say, at which point $f(Z_0;K) = 0$ (from equations (5) and (12)). Thus, the technique used for the accurate location of the point $Z = Z_0$ was as follows. Integration of equations (18) proceeded at regular intervals in Z until $f(Z;K) < 0$ (the point 1 in Fig. 1). However, the values of the dependent and independent variables had been retained at the preceding integration interval (the point 2 in Fig. 1). The equations (18) were then re-cast with f as the independent variable, i.e.

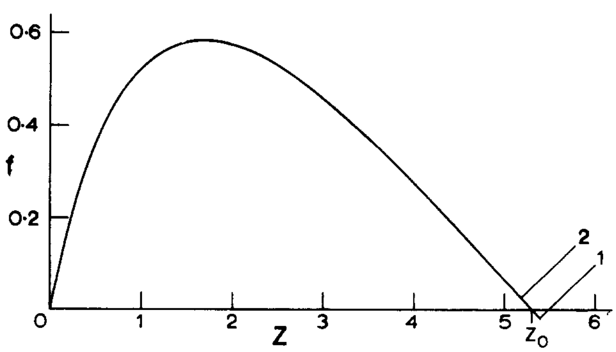

Figure 1. Graph of f against Z for $K = 1.5$.

$$\frac{dZ}{df}=\frac{1}{U}, \quad \frac{dg}{df}=\frac{V}{U}, \quad \frac{dh}{df}=\frac{f}{ZU},$$
$$\frac{dU}{df}=\frac{1}{UZ^2}(f^2-g^2-2ZhU), \quad \frac{dV}{df}=\frac{2}{UZ^2}(fg-ZhV). \tag{20}$$

Integration of the equations (20) was then recommenced from the point 2 until $f(Z_0; K) = 0$ was reached. Thus, the values of Z_0, $g(Z_0; K)$, $h(Z_0; K)$, $U(Z_0; K)$ and $V(Z_0; K)$ were obtained. It follows from equations (11) and (12) that

$$a_1 = c^2 Z_0, \quad b_1 = a_1/K, \quad c = (g(Z_0; K))^{-\frac{1}{4}},$$
$$A = -c(2h(Z_0; K) - 1), \quad F'(0) = -c^3 Z_0 U(Z_0; K), \quad G'(0) = -c^3 Z_0 V(Z_0; K). \tag{21}$$

The velocity profiles of $F(\zeta)$, $G(\zeta)$ and $H(\zeta)$, if required, could then be obtained from equations (12) since the third of the equations (11) gives

$$\zeta = \frac{1}{c}\ln(Z_0/Z). \tag{22}$$

Results obtained by this method for a range of values of K are given in Table 4. Confirmation of the results given in this table is provided by the series method based on equations (14) for those values of K for which the series contained in equations (14) converge. As we shall see, this restricts the possibility of confirmation to the cases $0 < K < 1.0701311$. The values of the various parameters obtained from equations (14), enclosed in brackets, are included for comparison in Table 4. They confirm the accuracy of the integration method at least for these values of K.

Equations (20) also provide a useful means of locating the nearest singularity in the series (12). Since the coefficients A_n and B_n in the latter series usually oscillate in sign, it follows that a singularity exists somewhere in the negative Z range. Thus, we presume that a singularity exists at $Z = -R$ ($R \geqslant 0$), where R is the radius of convergence of the series (12). The method used for the evaluation of R for a specified value of K was as follows. Equations (20) were integrated in terms of f from $f = 0$ through the negative f range until a large negative value of f had been achieved (usually about -10^{20}), at which point Z approached the limit $-R$. Note, once more, that this integration can be performed as an initial value problem, using the boundary conditions given in equations (18) and (19) at $f = 0$. From equations (18) it is possible to show that, as $Z \to -R$, $f \sim (Z+R)^{-2}$, $g \sim (Z+R)^{-2}$ and, in particular, $h \sim -3R/(Z+R)$. Thus, at each stage of the integration, $-hZ/(3+h)$ ($\simeq R$) was calculated and was found to approach R rather faster than $-Z$ approached that limit. By the above means, values of R were established for the values of K indicated in Table 4 and the results for R are included in that table. In order to provide some confirmation of the values of R obtained by this integral procedure, the ratios A_{n-1}/A_n for n large were calculated for those values of K for which the series contained in equations (14) converge, i.e. $K = 0.25, 0.5, 0.75, 1$. Unfortunately, it was found that the convergence of A_{n-1}/A_n was extremely slow and provided a poor estimate for R. However, since $f \sim (Z+R)^{-2}$, the method of Domb and Sykes [13] is useful in dealing with this problem. In this method the

TABLE 4

Results for A, a_1, b_1, c, Z_0, R, $F'(0)$, $G'(0)$ *obtained from equations* (18) *and* (20).
(*Values in brackets indicate comparative values obtained from equations* (14).)

K	A	a_1	b_1	c	Z_0	R	$F'(0)$	$-G'(0)$
0.25	1.25604266	0.25521406	1.02085625	1.43104487	0.12462297	1.09510937	0.34575619	1.37331065
	(1.2560426629	0.2552140612	1.0208562449	1.4310448731	0.1246229692		0.3457561864	1.3733106467)
0.5	0.56305869	0.54195317	1.08390634	1.04493269	0.49634669	1.73035182	0.45861422	0.89068691
	(0.5630586887	0.5419531711	1.0839063423	1.0449326934	0.4963466866		0.4586142197	0.8906869086)
0.75	$0.38492067\cdot10^{-1}$	0.89387922	1.19183896	0.89240195	1.12242627	2.10212739	0.50835010	0.63197420
	$(0.3849206650\cdot10^{-1}$	0.8938792214	1.1918389619	0.8924019474	1.1224262718		0.5083501023	0.6319741954)
1	−0.45255508	1.35218098	1.35218098	0.81289846	2.04626665	2.32563722	0.51561210	0.45135248
	(−0.4525550769	1.3521809839	1.3521809839	0.8128984566	2.0462666494		0.5156120968	0.4513524789)
1.07013112	−0.58918042	1.50718294	1.40840960	0.79723163	2.37135692	2.37135692	0.51188871	0.40947038
1.5	−1.44264216	2.85262963	1.90175309	0.73311313	5.30766365	2.55619531	0.45270694	0.21396126
2	−2.51401100	6.02925294	3.01462647	0.68938013	$1.26866214\cdot10^{1}$	2.66109159	0.35026814	$0.89056904\cdot10^{-1}$
3	−5.01188633	$3.08246972\cdot10^{1}$	$1.02748991\cdot10^{1}$	0.62142828	$7.98209315\cdot10^{1}$	2.74750511	0.19711927	$0.15365504\cdot10^{-1}$
4	−7.95428303	$1.76184271\cdot10^{2}$	$4.40460677\cdot10^{1}$	0.56226893	$5.57287070\cdot10^{2}$	2.78053228	0.12546924	$0.39523698\cdot10^{-2}$
5	$-1.12807991\cdot10^{1}$	$1.05552083\cdot10^{3}$	$2.11104166\cdot10^{2}$	0.51381283	$3.99813003\cdot10^{3}$	2.79637192	$0.88602500\cdot10^{-1}$	$0.13913022\cdot10^{-2}$
6	$-1.49505022\cdot10^{1}$	$6.48943823\cdot10^{3}$	$1.08157304\cdot10^{3}$	0.47466017	$2.88032516\cdot10^{4}$	2.80513063	$0.66876682\cdot10^{-1}$	$0.59823459\cdot10^{-3}$
7	$-1.89340007\cdot10^{1}$	$4.05912275\cdot10^{4}$	$5.79874678\cdot10^{3}$	0.44264783	$2.07164582\cdot10^{5}$	2.81046584	$0.52811753\cdot10^{-1}$	$0.29459712\cdot10^{-3}$
8	$-2.32085839\cdot10^{1}$	$2.57135874\cdot10^{5}$	$3.21419843\cdot10^{4}$	0.41602397	$1.48568238\cdot10^{6}$	2.81395068	$0.43086318\cdot10^{-1}$	$0.15997464\cdot10^{-3}$
9	$-2.77559120\cdot10^{1}$	$1.64505618\cdot10^{6}$	$1.82784020\cdot10^{5}$	0.39351109	$1.06234795\cdot10^{7}$	2.81635002	$0.36027875\cdot10^{-1}$	$0.93529237\cdot10^{-4}$
10	$-3.25607362\cdot10^{1}$	$1.06087153\cdot10^{7}$	$1.06087153\cdot10^{6}$	0.37419059	$7.57664729\cdot10^{7}$	2.81807134	$0.30711618\cdot10^{-1}$	$0.57934710\cdot10^{-4}$
∞						2.82545783		

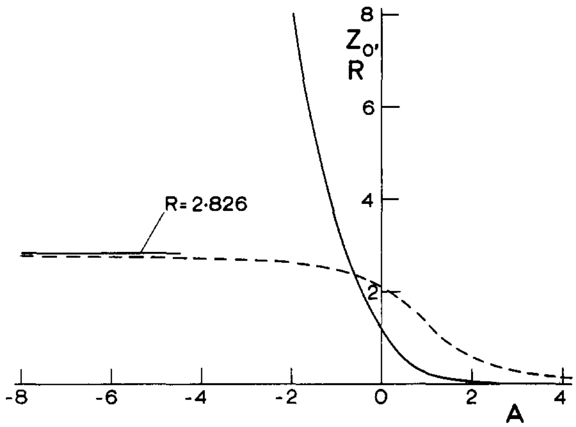

Figure 2. Graphs of Z_0, R against A.

——— Z_0, - - - - - R.

variation of A_n/A_{n-1} with $1/n$ is examined, the intercept at $1/n = 0$ giving a good estimate of R^{-1}. In fact, the method confirmed the first four figures of the values of R for $K = 0.25, 0.5, 0.75$ and 1 quoted in Table 4.

The series (12) provide uniformly valid solutions to equations (3) provided that $Z_0 < R$. It was deemed worthwhile to determine the values of K (and therefore A) at which $Z_0 = R$. The results given in Table 4 indicate that this condition occurs between $K = 1$ and $K = 1.5$. A simple iterative procedure, using Newton's method, established that $Z_0 = R = 2.3713569$ when $K = 1.0701311$, at which value $A = -0.5891804$. On the other hand, for all positive values of A, the results given in Section 3 and in Table 4 suggest that $Z_0 \leqslant R$. Consequently, we can conclude that the series (12) provide uniformly valid solutions to equations (3) within the range

$$-0.5891804 < A < +\infty. \tag{23}$$

The behaviour of Z_0 and R with A is illustrated in Fig. 2.

Clearly, for $A \leqslant -0.5891804$, the series method based directly on either equations (10) or (14) for the determination of a_1, b_1 and c becomes useless. However, even under these circumstances the series (7) and (12) are still convergent in the range $Z < R$. It is for this reason that the recurrence relations (13) can be used to determine the forms of the boundary conditions (18) and (19) for the differential equations (18) at $Z = 0$. The results given in Table 4 suggest that as $K \to \infty$ (at which limit it is presumed that $A \to -\infty$) R approaches a finite limit. In order to determine this limit, equations (20) have been used as described earlier in order to determine R for the case $1/K = 0$. The resulting value of R was found to be $R = 2.82545783$ and this result is included in Table 4 and in Fig. 2. Kuiken's [5] results confirm that as $A \to -\infty$ the series (7) and (12) remain valid in a finite region close to $Z = 0$; his results at the outer edge of the outer viscous region exhibit an exponential series form. The question then arises as to the forms the solutions take in the range $R \leqslant Z \leqslant Z_0$. Some clues, perhaps, can be gained from the rather peculiar behaviour of the coefficients B_n

(or b_n) of the series. As we have noted earlier, the coefficients A_n and B_n usually oscillate in sign for successive values of n. However, as Benton [11] has noted for the case $A = 0$, the coefficients B_{23} and B_{24} have the same sign, both being negative. Furthermore, we have found that B_{253} and B_{254} are both positive. No such behaviour in the coefficients A_n has been detected even though we have, on occasions, had to calculate up to A_{1000}. It should be noted that the values of n at which this behaviour in B_n has occurred do not appear to vary with A. For most of the values of A used, we have had to obtain B_{23} and B_{24} and this lack of a dependence on A is quite consistent. As for the coefficients B_{253} and B_{254}, these have had to be determined for the cases $A = -0.5, -0.55$ only but, once again, no dependence on A is indicated. It may be that this rather odd behaviour on the part of the coefficients B_n may indicate the presence of further singularities, perhaps of a complex nature, in the series (7) and (12). As an indication of the possibly complex nature of these additional singularities, it has been noted that the form of the inner solution obtained by Kuiken [5] for $A \to -\infty$, in the region of the disc, consists largely of trigonometric functions.

5. Conclusions

A series method has been discussed which provides solutions of high accuracy for the problem of steady rotating disc flow. It has been shown that the method covers all cases of surface suction and, in the limiting case, is in complete agreement with Stuart's [4] strong suction results. It has been shown further that low values of injection rate also can be dealt with by this method. For those injection cases for which the method fails, an alternative approach based on the method of Samuel and Hall [9] is successful. In this alternative approach, differential equations have been developed which describe the behaviour of the sums of the series both within and outside the region of convergence. Two advantages have been shown to accrue from this alternative approach. The first is that the solution for the flow can be determined as an initial value problem. The second advantage of the approach has been shown to be the ease with which the radius of convergence of the series can be obtained.

Once more, it is a pleasure to record the helpful advice of Dr. I. M. Hall.

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